

Isothermal Plasma Waves in Gravitomagnetic Planar Analogue

M. Sharif *and Umber Sheikh

Department of Mathematics, University of the Punjab,
Quaid-e-Azam Campus Lahore-54590, Pakistan.

Abstract

We investigate the wave properties of the Kerr black hole with isothermal plasma using 3+1 ADM formalism. The corresponding Fourier analyzed perturbed GRMHD equations are used to obtain the dispersion relations. These relations lead to the real values of the components of wave vector \mathbf{k} which are used to evaluate the quantities like phase and group velocities etc. These have been discussed graphically in the neighborhood of the pair production region. The results obtained verify the conclusion of Mackay et al. according to which rotation of a black hole is required for negative phase velocity propagation.

Keywords: 3+1 formalism, GRMHD, Kerr planar analogue, isothermal plasma.

1 Introduction

The study of gravitomagnetic waves in rotating black hole is important because the existence of black holes can be ultimately verified with the help of infalling plasma radiation and super-radiance of gravitomagnetic waves. Information from the magnetosphere can be transmitted from one region to another only by means of propagation allowed by plasma state. The gravitational field and the wind bring perturbations to the external fluid dynamics.

*e-mail: msharif@math.pu.edu.pk

The response of black holes to the external perturbations can be explored with the help of wave scattering method.

General Relativity is the theory of four-dimensional spacetime but we experience a three-dimensional space evolved in time. It is easier to split the spacetime into three-dimensional space and one-dimensional time to develop a better understanding of the physical phenomenon. The split we usually use in understanding the general relativistic physics of black holes and plasmas is 3+1 ADM split introduced by Arnowitt et al. [1]. This split in the formalization of general relativity is particularly appropriate for applications to the black hole theory as described by Thorne et al. [2]-[4]. Using this formalism, the wave propagation theory in the Friedmann universe was investigated by Holcomb and Tajima [5], Holcomb [6] and Dettmann et al. [7]. Komissarov [8] discussed the famous Blandford-Znajek solution.

Blandford and Znajek [9] found a process which describes the extraction of rotational energy in the form of Poynting flux. The black hole with a force free magnetosphere behaves as a battery with internal resistivity in a circuit made by poloidal current. This current system is considered to be equivalent to incoming and outgoing waves. The incoming waves transport energy in a direction opposite to the Poynting flux. Penrose [10] was the pioneer who gave the idea of extraction of energy from the rotating black hole by a specific process called Penrose process. In the wavelength analogue of Penrose process [11] an incoming wave with positive energy splits up into a transmitted wave with negative energy and a refracted wave with enhanced positive energy. The negative energy wave propagates into the black hole equivalent to a positive Poynting flux coming out of the horizon [12]-[13].

The key features of the Kerr black hole were beautifully summarized by Müller [14] who investigated the accretion physics in the plasma regime of the general relativistic magnetohydrodynamics (GRMHD). Punsley et al. [15] considered the black hole magnetohydrodynamics in a broader sense. Musil and Karas [16] observed the evolution of disturbances originated in outer parts of the accretion disk and developed a numerical scheme to show the transmission and reflection of waves. Koide et al. [17] modeled the GRMHD behavior of plasma flowing into rotating black hole in a magnetic field. They showed (numerical simulations) that energy of the spinning black hole can be extracted magnetically. Zhang [18]-[19] formulated the black hole theory for stationary symmetric GRMHD with its applications in Kerr geometry. He discussed wave modes for cold plasma with specific interface conditions. Buzzi et al. [20]-[21] provided a linearized treatment of transverse and lon-

gitudinal plasma waves in general relativistic two component plasma (3+1 ADM formalism) propagating in radial direction close to the Schwarzschild horizon.

Mackey et al. [22] gave the idea that negative phase velocity plane wave propagates in the ergosphere of a rotating black hole. They verify that the rotation of a black hole is required for negative phase velocity propagation which is a characteristic of Veselago medium. This medium was hypothetically mentioned by Veselago [23] and later formed experimentally [24] as a material (called metamaterial or left-handed material). Much work has been carried out to investigate the characteristics of this medium [25]. Woodley and Mojahedi [26] showed (using full wave simulations and analytical techniques) that in left-handed materials, the group velocity can be either positive (backwards wave propagation) or negative. Sharif and Umber [27]-[28] investigated some properties of plasma waves by investigating real wave numbers. The analysis has been done for the cold as well as isothermal plasmas living in the neighborhood of the event horizon by using Rindler approximation of the Schwarzschild spacetime. In a recent paper, the same authors [29] have found some interesting properties of cold plasma waves using perturbation wave analysis to the GRMHD equations in the vicinity of the Kerr black hole. They have also discussed the existence of Veselago medium near the pair production region.

This paper has been extended to investigate the wave properties for the isothermal plasma. We have focussed this work to investigate the following three main objectives:

1. The behavior of gravitomagnetic waves under the influence of gravity and magnetospheric wind is analysed. This helps us to detect the response of the black hole magnetospheric plasma oscillations to gravitomagnetic perturbations near the pair production region. The pair production region lies near the event horizon of the black hole.
2. The existence of Veselago medium in the black hole regime is checked.
3. The negative phase velocity propagation regions are investigated and compare the results with those obtained by Mackay et al. [22].

To this end, we derive the GRMHD equations in 3+1 formalism using the isothermal equation of state. The component form of the equations for specific background assumptions is obtained by using perturbations. We consider the perturbed quantities as plane harmonic waves produced by gravity

and wind due to black hole rotation. The Fourier analysis method for waves is applied and dispersion relations are derived. These relations lead to the x -component of the wave vector from which the relevant quantities are investigated to analyze the wave properties near the pair production region.

The paper is organized as follows. The next section is oriented with the description of the Kerr analogue spacetime and mathematical laws in 3+1 formalism for this model. Section **3** is devoted to the assumptions corresponding to the background flow. In section **4**, the GRMHD equations along with their Fourier analyzed perturbed form for the isothermal equation of state of plasma are given. Section **5** provides the solutions of dispersion relations. In the last section, we shall discuss the results.

2 Mathematical Framework

This section contains the line element for a general spacetime model. The electrodynamics corresponding to Kerr planar analogue in 3+1 formalism is also considered.

2.1 Description of Model Spacetime

The line element of the spacetime in 3+1 formalism can be written as

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \quad (2.1)$$

where lapse function (α), shift vector (β) and spatial metric (γ) are functions of time and space coordinates.

We consider the planar analogue of Kerr spacetime [19], i.e.,

$$ds^2 = -dt^2 + (dx + \beta(z)dt)^2 + dy^2 + dz^2. \quad (2.2)$$

Here z , x , y and t correspond to Kerr's radial r , axial ϕ , poloidal θ and time t coordinates. The Kerr metric depends non-trivially on both r and θ , whereas this model metric depends on z only. The lapse function α is taken to be unity to avoid the effects of horizon and redshift. The value of the shift function β (analogue to the Kerr-type gravitomagnetic potential) decreases monotonically from 0 ($z \rightarrow \infty$) to some constant value ($z \rightarrow -\infty$). We have assumed the direction of β along x -axis. This shift function derives an MHD wind which extracts translational energy analogous to the rotational

energy for the Kerr metric. The shift vector in three dimensions will be denoted by the Greek letter β . The Kerr-type horizon has been pushed off to $z = -\infty$. The pair production region lies at $z = 0$ where the plasma is created. The newly created particles are then driven up to relativistic velocities by magnetic-gravitomagnetic coupling as they flow off to infinity and down towards the horizon. Geometrized units will be used throughout the paper.

2.2 Electrodynamics in Kerr Planar Analogue

We consider the magnetosphere filled with MHD fluid and take the perfect MHD flow condition in fluid's rest-frame

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0 \quad (2.3)$$

with \mathbf{V} , \mathbf{B} and \mathbf{E} are fiducial observer (FIDO) measured fluid velocity, magnetic and electric fields respectively. For perfect MHD flow in (2.1) with $\alpha = 1$, differential form of Faraday's law in 3+1 formalism [18] turn out to be

$$\frac{d\mathbf{B}}{d\tau} + (\mathbf{B} \cdot \nabla)\beta - (\nabla \cdot \beta)\mathbf{B} = \nabla \times (\mathbf{V} \times \mathbf{B}), \quad (2.4)$$

where $\frac{d}{d\tau} \equiv \frac{\partial}{\partial t} - \beta \cdot \nabla$ is the FIDO measured rate of change of any three-dimensional vector in absolute space. Gauss law of magnetism according to FIDO can be written as [18]

$$\nabla \cdot \mathbf{B} = 0. \quad (2.5)$$

For (2.1) with $\alpha = 1$, the local conservation law of rest-mass [18] according to FIDO is

$$\frac{D\rho_0}{D\tau} + \rho_0 \gamma^2 \mathbf{V} \cdot \frac{D\mathbf{V}}{D\tau} + \rho_0 \nabla \cdot (\mathbf{V} - \beta) = 0, \quad (2.6)$$

where ρ_0 is the rest-mass density, γ is the Lorentz factor and $\frac{D}{D\tau} \equiv \frac{d}{d\tau} + \mathbf{V} \cdot \nabla = \frac{\partial}{\partial t} + (\mathbf{V} - \beta) \cdot \nabla$ is the time derivative moving along the fluid. The FIDO measured law of force balance equation [18] for the spacetime, given

by Eq.(2.1) with $\alpha = 1$, takes the form

$$\begin{aligned}
& \left\{ \left(\rho_0 \gamma^2 \mu + \frac{\mathbf{B}^2}{4\pi} \right) \gamma_{ij} + \rho_0 \gamma^4 \mu V_i V_j - \frac{1}{4\pi} B_i B_j \right\} \frac{DV^j}{D\tau} + \rho_0 \gamma^2 V_i \frac{D\mu}{D\tau} \\
& - \left(\frac{\mathbf{B}^2}{4\pi} \gamma_{ij} - \frac{1}{4\pi} B_i B_j \right) V^j{}_{,k} V^k = -\rho_0 \gamma^2 \mu \beta_{j,i} V^j - p_{,i} \\
& + \frac{1}{4\pi} (\mathbf{V} \times \mathbf{B})_i \nabla \cdot (\mathbf{V} \times \mathbf{B}) - \frac{1}{8\pi} (\mathbf{B})^2_{,i} + \frac{1}{4\pi} B_{i,j} B^j \\
& - \frac{1}{4\pi} [\mathbf{B} \times \{ \mathbf{V} \times (\nabla \times (\mathbf{V} \times \mathbf{B}) - (\mathbf{B} \cdot \nabla) \beta) + (\mathbf{V} \times \mathbf{B}) \cdot \nabla \beta \}]_i, \quad (2.7)
\end{aligned}$$

where μ is the specific enthalpy and p is the pressure of the fluid. The FIDO measured local energy conservation law (Eq.(2.4) of [28]), for Eq.(2.1) with $\alpha = 1$, is given as follows

$$\begin{aligned}
& \rho_0 \gamma^2 \frac{D\mu}{D\tau} + \mu \gamma^2 \frac{D\rho_0}{D\tau} + 2\rho_0 \mu \gamma^4 \mathbf{V} \cdot \frac{D\mathbf{V}}{D\tau} - \frac{dp}{d\tau} - \mu \rho_0 \gamma^2 \nabla \cdot \beta \\
& + \rho_0 \mu \gamma^2 (\nabla \cdot \mathbf{V}) - \rho_0 \mu \gamma^2 \mathbf{V} \cdot (\mathbf{V} \cdot \nabla) \beta + \frac{1}{4\pi} \{ (\mathbf{V} \times \mathbf{B}) \cdot (\nabla \times \mathbf{B}) \\
& + (\mathbf{V} \times \mathbf{B}) \cdot \frac{d}{d\tau} (\mathbf{V} \times \mathbf{B}) + (\mathbf{V} \times \mathbf{B}) \cdot \{ (\mathbf{V} \times \mathbf{B}) \cdot \nabla \} \beta \\
& - (\mathbf{V} \times \mathbf{B}) \cdot (\mathbf{V} \times \mathbf{B}) (\nabla \cdot \beta) \} = 0. \quad (2.8)
\end{aligned}$$

Eqs.(2.4)-(2.8) give the perfect GRMHD equations.

3 Specialization of Background Flow for Model Spacetime

In this section, we give the background flow and relative assumptions which will be used to simplify the problem.

3.1 Description of Flow Quantities

The FIDO measured 4-velocity of fluid can be described by a spatial vector field lying in the xz -plane [19]

$$\mathbf{V} = V(z) \mathbf{e}_x + u(z) \mathbf{e}_z.$$

Here the Lorentz factor takes the form $\gamma = \frac{1}{\sqrt{1-u^2-V^2}}$. The magnetic field measured by FIDO is also assumed to be in the xz -direction

$$\mathbf{B} = B\{\lambda(z)\mathbf{e}_x + \mathbf{e}_z\},$$

where B is constant. The corresponding Poynting vector becomes

$$\mathbf{S} = \frac{1}{4\pi}(\mathbf{E} \times \mathbf{B}).$$

We have considered an example of stationary flow of an isothermal MHD fluid in our model spacetime (2.2). These flows are used as stationary model magnetospheres whose dynamical perturbations are to be studied. The plasma is moving in the xz -direction. The perturbed flow is along z -direction due to the black hole's gravity and along x -direction due to rotation of the black hole (in direction of shift vector of our analogue spacetime). This flow will be analyzed to seek the properties of plasma waves.

3.2 Perturbations and Wave Propagation

The perturbed flow in the magnetosphere (which is in the xz -plane) can be characterized by velocity \mathbf{V} , magnetic field \mathbf{B} , the fluid density ρ and pressure p . We denote the unperturbed quantities by a superscript zero and the following dimensionless notations are used for perturbations ($\delta\mathbf{V}$, $\delta\mathbf{B}$, $\delta\rho$, δp)

$$\begin{aligned}\tilde{\rho} &\equiv \frac{\delta\rho}{\rho} = \tilde{\rho}(t, x, z), & \tilde{p} &\equiv \frac{\delta p}{p} = \tilde{p}(t, x, z), \\ \mathbf{v} &\equiv \delta\mathbf{V} = v_x(t, x, z)\mathbf{e}_x + v_z(t, x, z)\mathbf{e}_z, \\ \mathbf{b} &\equiv \frac{\delta\mathbf{B}}{B} = b_x(t, x, z)\mathbf{e}_x + b_z(t, x, z)\mathbf{e}_z.\end{aligned}\tag{3.1}$$

The perturbed variables take the following form

$$\begin{aligned}\rho &= \rho^0 + \rho\tilde{\rho}, & p &= p^0 + p\tilde{p}, \\ \mathbf{V} &= \mathbf{V}^0 + \mathbf{v}, & \mathbf{B} &= \mathbf{B}^0 + B\mathbf{b}.\end{aligned}\tag{3.2}$$

It is also assumed that the perturbations have sinusoidal dependence of t , x and z . Thus

$$\begin{aligned}\tilde{\rho}(t, x, z) &= c_1 e^{-i(\omega t - k_x x - k_z z)}, & \tilde{p}(t, x, z) &= c_6 e^{-i(\omega t - k_x x - k_z z)}, \\ v_x(t, x, z) &= c_2 e^{-i(\omega t - k_x x - k_z z)}, & v_z(t, x, z) &= c_3 e^{-i(\omega t - k_x x - k_z z)}, \\ b_x(t, x, z) &= c_4 e^{-i(\omega t - k_x x - k_z z)}, & b_z(t, x, z) &= c_5 e^{-i(\omega t - k_x x - k_z z)}.\end{aligned}\tag{3.3}$$

Using the values of components of \mathbf{k} , we can discuss the quantities like phase velocity vector, group velocity vector, refractive index and its change with respect to angular frequency. These quantities would help to investigate the wave behavior of the Kerr black hole magnetosphere and the properties of Veselago medium.

4 GRMHD Equations for Kerr Spacetime in Isothermal State of Plasma

The isothermal equation of state means that there is no exchange of energy between the plasma and the magnetic field. This state can be expressed by the following equation

$$\mu = \frac{\rho + p}{\rho_0} = \text{constant}.$$

When we use this equation of state, the set of GRMHD Eqs.(2.4)-(2.8) take the following form for the spacetime given in Eq.(2.2), i.e, $\beta = (\beta_x, 0, 0)$

$$\frac{d\mathbf{B}}{d\tau} + (\mathbf{B} \cdot \nabla)\beta = \nabla \times (\mathbf{V} \times \mathbf{B}), \quad (4.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (4.2)$$

$$\frac{D(\rho + p)}{D\tau} + (\rho + p)\gamma^2 \mathbf{V} \cdot \frac{D\mathbf{V}}{D\tau} + (\rho + p)\nabla \cdot \mathbf{V} = 0, \quad (4.3)$$

$$\begin{aligned} & \left[\left\{ (\rho + p)\gamma^2 + \frac{\mathbf{B}^2}{4\pi} \right\} \delta_{ij} + (\rho + p)\gamma^4 V_i V_j - \frac{1}{4\pi} B_i B_j \right] \frac{dV^j}{d\tau} \\ & + (\rho + p)\gamma^2 V_{i,k} V^k + (\rho + p)\gamma^4 V_i V_j V^j_{,k} V^k = (\rho + p)\gamma^2 \beta_{j,i} V^j \\ & - p_{,i} + \frac{1}{4\pi} (B_{i,j} - B_{j,i}) B^j - \frac{1}{4\pi} \left\{ \mathbf{B} \times \left(\mathbf{V} \times \frac{d\mathbf{B}}{d\tau} \right) \right\}_i, \end{aligned} \quad (4.4)$$

$$\begin{aligned} & \gamma^2 \mathbf{V} \cdot \frac{D}{D\tau} (\rho + p) + 2(\rho + p)\gamma^4 \mathbf{V} \cdot \frac{D\mathbf{V}}{D\tau} - \frac{dp}{d\tau} + (\rho + p)\gamma^2 (\nabla \cdot \mathbf{V}) \\ & - (\rho + p)\gamma^2 \mathbf{V} \cdot (\mathbf{V} \cdot \nabla)\beta + \frac{1}{4\pi} [(\mathbf{V} \times \mathbf{B}) \cdot (\nabla \times \mathbf{B}) \\ & + (\mathbf{V} \times \mathbf{B}) \cdot \frac{d}{d\tau} (\mathbf{V} \times \mathbf{B}) + (\mathbf{V} \times \mathbf{B}) \cdot \{(\mathbf{V} \times \mathbf{B}) \cdot \nabla\} \beta] = 0. \end{aligned} \quad (4.5)$$

These equations proceed in a similar way as used in [27]-[28]. Equations (3.1) and (3.2) as well as the restrictions for the velocity and magnetic fields,

given in Section 3.1, lead to the perturbed form of Eqs.(4.1)-(4.5) given in Appendix A. When we use Eq.(3.3), the Fourier analyzed perturbed equations take the following form

$$\begin{aligned} & \iota k_z c_2 - (\iota k_z \lambda + \lambda') c_3 - c_4 (\iota k_z u - \iota \omega + u') \\ & + c_5 \{ (V - \beta) \iota k_z + (V - \beta)' \} = 0, \end{aligned} \quad (4.6)$$

$$\iota k_x c_2 - \iota k_x \lambda c_3 + \iota c_5 \{ (V - \beta) k_x + k_z u - \omega \} = 0, \quad (4.7)$$

$$k_x c_4 = -k_z c_5, \quad (4.8)$$

$$\begin{aligned} & c_1 [\iota \rho \{ -\omega + (V - \beta) k_x + u k_z \} - \{ p' u + p u' + p u \gamma^2 (V V' + u u') \}] \\ & + c_2 (\rho + p) [-\iota (\omega + \beta k_x) \gamma^2 V + \iota k_z \gamma^2 u V + \iota k_x (1 + \gamma^2 V^2) \\ & + \gamma^2 u \{ (1 + 2\gamma^2 V^2) V' + 2\gamma^2 u V u' \}] + c_3 (\rho + p) [-\iota (\omega + \beta k_x) \gamma^2 u + \iota k_x \gamma^2 u V \\ & + \iota k_z (1 + \gamma^2 u^2) - (1 - 2\gamma^2 u^2) (1 + \gamma^2 u^2) \frac{u'}{u} + 2\gamma^4 u^2 V V'] + c_6 [\iota p \{ -\omega \\ & + (V - \beta) k_x + u k_z \} + \{ p' u + p u' + p u \gamma^2 (V V' + u u') \}] = 0, \quad (4.9) \\ & c_1 \rho \gamma^2 u \{ (1 + \gamma^2 V^2) V' + \gamma^2 u V u' \} + c_6 p [\gamma^2 u \{ (1 + \gamma^2 V^2) V' + \gamma^2 u V u' \} + \iota k_x] \\ & + c_2 \left[-\iota (\omega + \beta k_x) \left\{ (\rho + p) \gamma^2 (1 + \gamma^2 V^2) + \frac{B^2}{4\pi} \right\} \right. \\ & + \iota (k_x V + k_z u) \left\{ (\rho + p) \gamma^2 (1 + \gamma^2 V^2) - \frac{B^2}{4\pi} \right\} + (\rho + p) \gamma^4 u \{ (1 + 4\gamma^2 V^2) u u' \\ & + 4(1 + \gamma^2 V^2) V V' \}] + c_3 \left[-\iota (\omega + \beta k_x) \left\{ (\rho + p) \gamma^4 u V - \frac{\lambda B^2}{4\pi} \right\} \right. \\ & + \iota (k_x V + k_z u) \left\{ (\rho + p) \gamma^4 u V + \frac{\lambda B^2}{4\pi} \right\} + (\rho + p) \gamma^2 [2\gamma^2 (1 + 2\gamma^2 u^2) u V u' \\ & + \{ (1 + 2\gamma^2 u^2) (1 + 2\gamma^2 V^2) - \gamma^2 V^2 \} V'] + \frac{B^2 u \lambda'}{4\pi}] \\ & + \frac{B^2}{4\pi} c_4 \{ -\iota k_z (1 - u^2) + u u' \} + \frac{B^2}{4\pi} c_5 \{ -\lambda' - u (V - \beta)' + \iota k_x (1 - V^2) \\ & - 2\iota u V k_z \} = 0, \end{aligned} \quad (4.10)$$

$$\begin{aligned} & c_1 \gamma^2 \rho [u \{ (1 + \gamma^2 u^2) u' + \gamma^2 V u V' \} - V \beta'] \\ & + c_6 [\gamma^2 p \{ (1 + \gamma^2 u^2) u u' + \gamma^2 V u^2 V' - V \beta' \} + p' + p \iota k_z] \\ & + c_2 \left[-\iota (\omega + \beta k_x) \left\{ (\rho + p) \gamma^4 u V - \frac{\lambda B^2}{4\pi} \right\} + \iota (k_x V + k_z u) \left\{ (\rho + p) \gamma^4 u V \right. \right. \\ & \left. \left. + \frac{\lambda B^2}{4\pi} \right\} + (\rho + p) \gamma^2 \{ \gamma^2 u^2 V' (1 + 4\gamma^2 V^2) - \beta' (1 + 2\gamma^2 V^2) \} \right] \end{aligned}$$

$$\begin{aligned}
& +2V\gamma^2uu'(1+2\gamma^2u^2)\} + c_3 [-\iota(\omega + \beta k_x) \{(\rho + p)\gamma^2(1 + \gamma^2u^2) \\
& + \frac{\lambda^2 B^2}{4\pi}\} + \iota(k_x V + k_z u) \left\{(\rho + p)\gamma^2(1 + \gamma^2u^2) - \frac{\lambda^2 B^2}{4\pi}\right\} \\
& + (\rho + p)\gamma^2[u'(1 + \gamma^2u^2)(1 + 4\gamma^2u^2) + 2u\gamma^2\{(1 + 2\gamma^2u^2)VV' - V\beta'\}] \\
& - \frac{B^2}{4\pi}\lambda\lambda'u] + \frac{B^2}{4\pi}c_4 \{\iota k_z \lambda(1 - u^2) + \lambda' - \lambda uu'\} + \frac{B^2}{4\pi}c_5 \{2\lambda u V \iota k_z \\
& + \lambda u(V - \beta)' - \lambda \iota k_x(1 - V^2)\} = 0, \tag{4.11} \\
& c_1 \gamma^2 [-\iota \omega \rho + \rho' u + \rho u' + 2\rho u \gamma^2(VV' + uu') - \rho \gamma^2 u V \beta' + \iota k_x \rho(V - \beta) \\
& + \rho \iota k_z u] + c_6 [-\iota \omega p(\gamma^2 - 1) + \gamma^2\{p' u + p u' + 2p u \gamma^2(VV' + uu') \\
& - p \gamma^2 u V \beta' + \iota k_x p(V - \beta) + p \iota k_z u\}] + c_2 [-\iota \omega \{2(\rho + p)\gamma^4 V \\
& - \frac{B^2}{4\pi}(u\lambda - V)\} + \iota k_x [(\rho + p)\gamma^2\{1 + 2\gamma^2 V(V - \beta)\} - \frac{B^2}{4\pi}(V - \beta)(u\lambda - V)] \\
& + \iota k_z u \{2(\rho + p)\gamma^4 V + \frac{B^2}{4\pi}(u\lambda - V)\} + (\rho + p)\gamma^2 u \{2\gamma^2 V' \\
& + 6\gamma^4 V(VV' + uu') - \beta'(1 + 2\gamma^2 V^2)\} - \frac{B^2 \lambda'}{4\pi}] + c_3 [-\iota \omega \{2(\rho + p)\gamma^4 u \\
& + \frac{\lambda B^2}{4\pi}(u\lambda - V)\} + \iota k_x (V - \beta) \{2(\rho + p)\gamma^4 u + \frac{B^2 \lambda}{4\pi}(u\lambda - V)\} \\
& + \iota k_z \{(\rho + p)\gamma^2(1 + 2\gamma^2 u^2) - \frac{B^2 \lambda u}{4\pi}(u\lambda - V)\} + (\rho + p)\gamma^2 \{-\frac{u'}{u} \\
& + 2\gamma^2 uu' + 6\gamma^4 u^2(VV' + uu') + \gamma^2(VV' + uu') - V\beta'(1 + \gamma^2 u^2)\} \\
& + \frac{B^2 \lambda'}{4\pi} \{\lambda - u(u\lambda - V)\}] + c_4 \frac{B^2}{4\pi} [u\{\lambda' - (u\lambda - V)u'\} \\
& + \iota k_z (u\lambda - V)(1 - u^2)] + c_5 \frac{B^2}{4\pi} \{-\lambda' V + u(u\lambda - V)(V - \beta)' \\
& - \iota k_x (u\lambda - V)(1 - V^2) + 2\iota k_z u V\} = 0. \tag{4.12}
\end{aligned}$$

Eq.(4.8) gives the relation between x and z -components of the wave vector i.e., $k_z = -\frac{c_4}{c_5}k_x$ which will be used in the next section.

5 Numerical Solutions

This section is devoted to the numerical solutions of the dispersion equations. The following subsection contains the relative assumptions which make the equations easier to deal with.

5.1 Relative Assumptions

In our stationary symmetric background, the x -component of the velocity vector can be written in the form [19] $V = C + \lambda u$, where $C \equiv \beta + V_F$ with V_F as an integration constant. We assume the value of the shift function [19] $\beta = \tanh(z) - 1$ with $V_F = 1$. Further, $B^2 = 8\pi$ and $\lambda = 1$ are taken so that the magnetic field becomes constant. Thus the x -component takes the form

$$V = 1 + \beta + u = \tanh(z) + u.$$

Substituting the value of V in the conservation law of rest-mass for three-dimensional hypersurface, i.e.,

$$\rho_0 \gamma u = \mu(\rho + p) \gamma u = A \text{ (constant)} \quad (5.1)$$

with the assumption that rest-mass density is constant, we get an equation of the form

$$3u^2 + 2u \tanh(z) + \tanh^2(z) - 1 = 0$$

quadratic in u with the assumption that $A/\rho_0 = 1$. The solutions of this equation and the corresponding values of V are given as follows

$$\begin{aligned} u_1 &= -\frac{1}{3} \tanh(z) - \frac{1}{3} \sqrt{3 - 2 \tanh^2(z)}, \\ V_1 &= \frac{2}{3} \tanh(z) - \frac{1}{3} \sqrt{3 - 2 \tanh^2(z)}, \end{aligned} \quad (5.2)$$

$$\begin{aligned} u_2 &= -\frac{1}{3} \tanh(z) + \frac{1}{3} \sqrt{3 - 2 \tanh^2(z)}, \\ V_2 &= \frac{2}{3} \tanh(z) + \frac{1}{3} \sqrt{3 - 2 \tanh^2(z)}. \end{aligned} \quad (5.3)$$

We shall use these values to solve the dispersion relations. The Poynting vector for these values takes the following form

$$\mathbf{S} = 2 \tanh(z) (\mathbf{e}_x - \mathbf{e}_z).$$

These quantities are valid for the region outside the pair production region. We consider the region $-5 \leq z \leq 5$ and omit the region $-1 \leq z \leq 1$ due to large variations in the background flow quantities. In the rest of the region, these quantities become constant and Fourier analyzed procedure is valid for this region. Further, we use the relation $k_z = -k_x$ which reduces the wave vector to $(k_x, 0, -k_x)$.

The computer programming (using *Mathematica*) is used to evaluate a root of the dispersion relation for the plasma moving towards the black hole with the velocity components given by Eq.(5.2). This is given as a separate file with all the required codes. Other roots can be obtained in a similar manner.

It is observed that the sextic equation has all roots admitting imaginary values at several points. The quintic equation gives one real root of the positive z region for both the values of velocity (Eqs.(5.2) and (5.3)) shown in the Figures 1 and 2. For the negative z region, the velocity, given by Eq.(5.2), gives one real root shown in the Figure 3. The approximated root becomes imaginary at $z = -5$ which we omit and our mesh reduces to $-4.8 \leq z \leq -1$, $0 \leq \omega \leq 10$ for the interpolating function. The values at $z = -5$ are extrapolated afterwards. The velocity components, given by Eq.(5.3) for negative z region, leads to three real roots shown in the Figures 4, 5 and 6. The real data values for the root give a real interpolation function.

It is clear that the Figures 1 and 2 represent the neighborhood of the pair production region towards the outer end (as the wave number is found for the positive values of z) whereas the Figures 3, 4, 5 and 6 show the neighborhood of the pair production region towards the event horizon (as the wave number is determined for the negative values of z).

5.2 Results

First, we obtain k_x for the velocity components, given by Eqs.(5.2) and (5.3) in the positive z region. These are shown in the Figures 1 and 2 respectively.

In the Figure 1, the x -component of the wave vector is negative near the pair production region and for the waves with negligible angular frequency. For each angular frequency, the waves grow monotonically in number when they move away from the event horizon. There is a sudden increase in the x -component of the wave vector, it admits positive values for a particular value of z , then decreases a bit and smoothly increases afterwards. The fluid near the pair production region (region with negative x -component of the wave vector) possesses the negative values for the x -components of the phase and group velocities. For this region, the wave vector is in the opposite direction of the Poynting vector and hence it shows the existence of Veselago medium there [23]. For the same region, the phase and group velocity vectors are in the direction opposite to the Poynting flux and hence the regions are of negative phase and group velocity propagation. The change in the

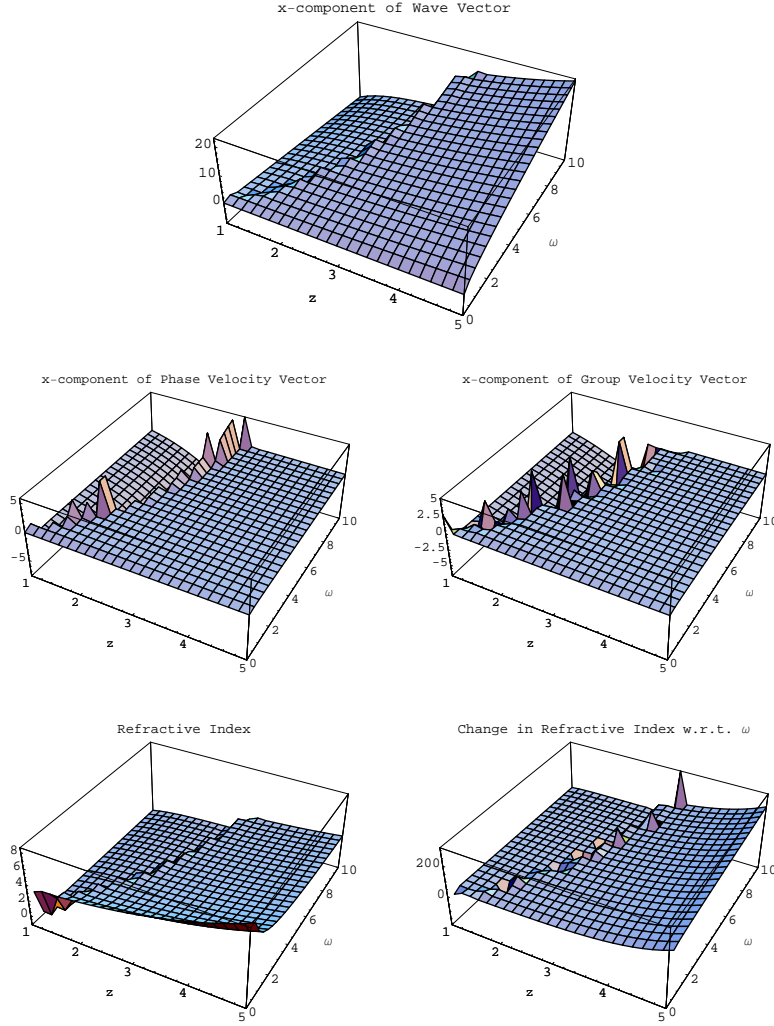


Figure 1: For the velocity components given by Eq.(5.2), plasma admits the properties of Veselago medium near the pair production region. As the waves move away from the pair production region, they disperse normally. Negative phase and group velocity propagation regions are observed near the pair production region.

refractive index with respect to the angular frequency is positive for the regions (i) $1.6 \leq z < 2$, $0.4 \leq \omega \leq 1$, (ii) $2 \leq z < 3$, $0.42 \leq \omega \leq 2.92$, (iii) $3 \leq z < 4$, $0.225 \leq \omega \leq 10$ and (iv) $4 \leq z \leq 5$, $0.14 \leq \omega \leq 10$ for which the dispersion is normal [26], [30]. In the rest of the region, most of the points admit anomalous dispersion.

The Figure 2 shows that the x -component of the wave vector is negative for the region $1.0 \leq z \leq 1.89$. It is large near the event horizon and decreases up to $z = 1.75$ after which it increases and fluctuations occur in the values. In the region $1.89 < z \leq 10$, it takes random values. The negative values of k_x in the region implies that the wave vector is in the opposite direction to the Poynting vector which indicates the properties of Veselago medium. In the region $0 \leq z \leq 1.89$, the x -components of the phase and group velocities take negative values and hence this is of negative phase and group velocity propagation region. Both these quantities admit random values in the region $1.89 \leq z \leq 10$. For the region $1 \leq z \leq 1.6$, $0 < \omega \leq 0.079$, the change in the refractive index with respect to the angular frequency is positive and hence the dispersion is found to be normal. In the region $1 \leq z \leq 1.6$, $0.079 < \omega \leq 10$, the quantity $\frac{dn}{d\omega} < 0$ which indicates anomalous dispersion in this region [26]. The rest of the region shows random points of normal as well as anomalous dispersion.

For the negative z region, i.e., the region towards the event horizon of the black hole in the neighborhood of the pair production region, we obtain one value of k_x for the velocity components given by Eq.(5.2) and three for the velocity components given by Eq.(5.3). These values are shown respectively by the Figures 3, 4, 5 and 6.

In the Figure 3, the x -component of the wave vector is negative for the region $-1.925 \leq z \leq -1.0$ where the Poynting vector is parallel to the wave vector and hence the medium is usual. The refractive index greater than one and positive change in the refractive index with respect to the angular frequency indicate normal dispersion. In the rest of the region, all the three quantities admit random values and hence there are normal as well as anomalous points of dispersion. For the region $-1.4 \leq z \leq -1.0$, $0.5 \leq \omega \leq 10$, the x -components of phase and group velocities are negative such that $v_{px} > v_{gx}$. These velocity components admit random values in rest of the region.

The Figure 4 indicates that the x -component of the wave vector is negative for the whole region and hence the wave vector and the Poynting vector are in the same direction showing the existence of the usual medium. As the

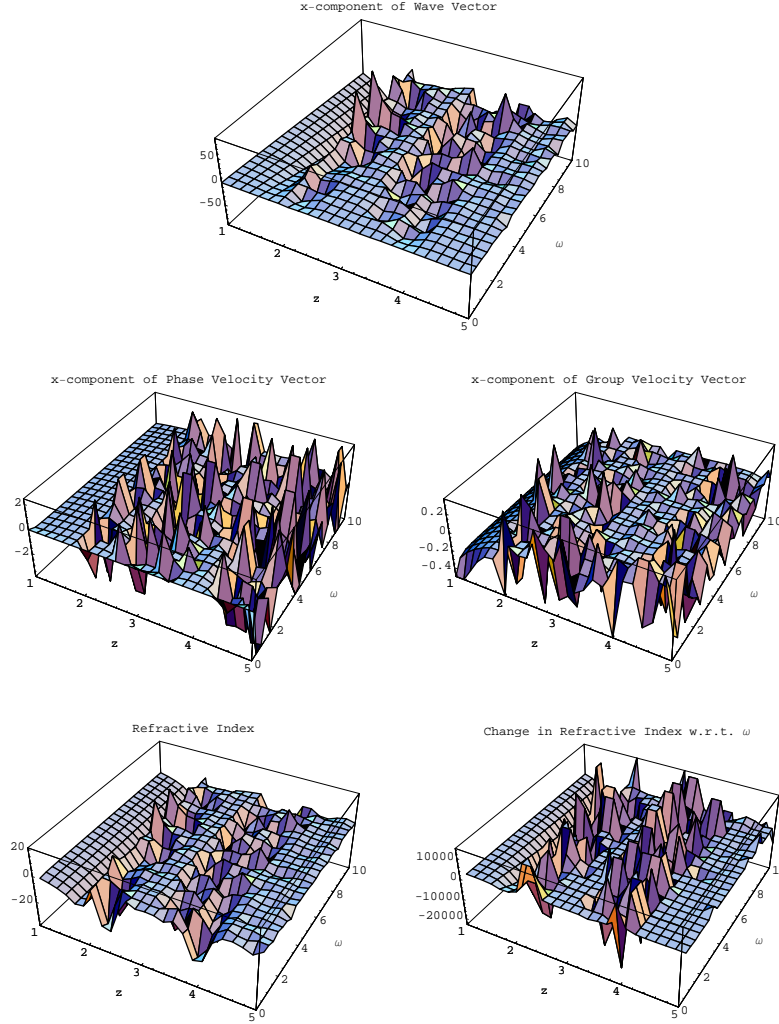


Figure 2: Veselago medium exists near the pair production region for the velocity components given by Eq.(5.3). In the same region, negative phase and group velocity propagation regions are observed. Most of the region in the neighborhood of the pair production region shows anomalous dispersion.

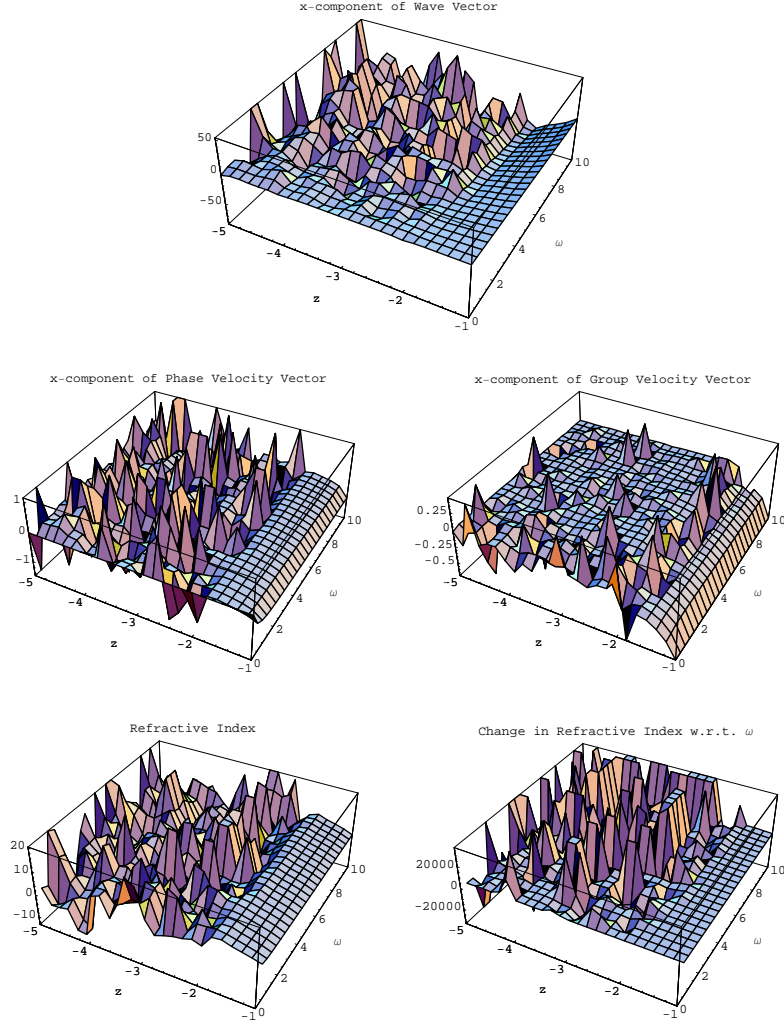


Figure 3: Near the pair production region, the dispersion is normal whereas the rest of the region admits normal as well as anomalous points of dispersion for velocity components given in Eq.(5.2).

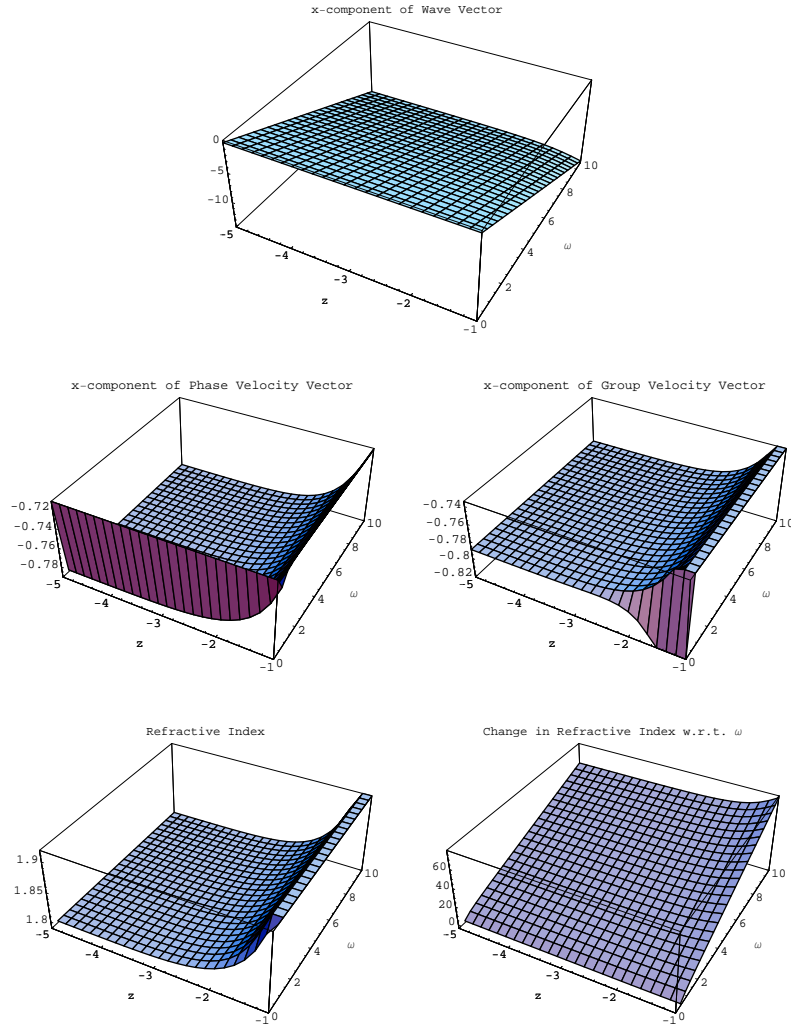


Figure 4: For the velocity components given by Eq.(5.3), dispersion is normal except for the waves with very low angular frequency.

values of z and ω grow, k_x decreases and hence v_{px} and v_{gx} are negative in this region. Although $v_{px} > v_{gx}$ for the region $0 \leq \omega \leq 10^{-15}$, yet they are nearly equal for rest of the region. The refractive index is greater than one in the whole region. The refraction increases as the waves move towards the pair production region. The change in the refractive index with respect to the angular frequency is negative for the region $0 < \omega \leq 0.6$ which shows that anomalous dispersion of waves. For the rest of the region, it is positive and thus indicates normal dispersion.

The Figure 5 shows that k_x is positive for the whole region. Thus the wave vector is in the opposite direction to the Poynting vector which indicates the presence of Veselago medium. The quantity k_x increases with the increase in z and ω except for the waves with negligible angular frequency. v_{px} and v_{gx} are nearly equal and admit positive values which show negative phase and group velocity propagation in the whole region. The refractive index is negative and decreases as the values of z and ω increase. The change in the refractive index with respect to the angular frequency is negative for the regions (i) $-2 \leq z \leq -1$, $2.15 \leq \omega \leq 10$ (ii) $-3 \leq z < -2$, $5.5 \leq \omega \leq 10$ and (iii) $-4 \leq z < -3$, $9.5 \leq \omega \leq 10$ which indicates anomalous dispersion in these regions. It is positive for the rest of the region which indicates normal dispersion except for the waves with negligible angular frequency.

In the Figure 6, the x -component of the wave vector is positive for the whole region and increases with the increase in the angular frequency. As the waves move away from the pair production region, the x -component of the phase velocity decreases slightly and then increases. In contrast, the x -component of the group velocity increases a little and then decreases. The refractive index is negative due to the fact that the Poynting vector is in the opposite direction to the wave vector which shows the existence of Veselago medium. The positive values of x -components of phase and group velocities show the existence of negative phase and group velocity propagation regions. The change in the refractive index with respect to the angular frequency is negative throughout the region and hence shows anomalous dispersion except for the waves with negligible angular frequency.

6 Conclusion

It is well-known that charged particles are created in the pair production region. Some of these particles which are pushed on to orbits with negative

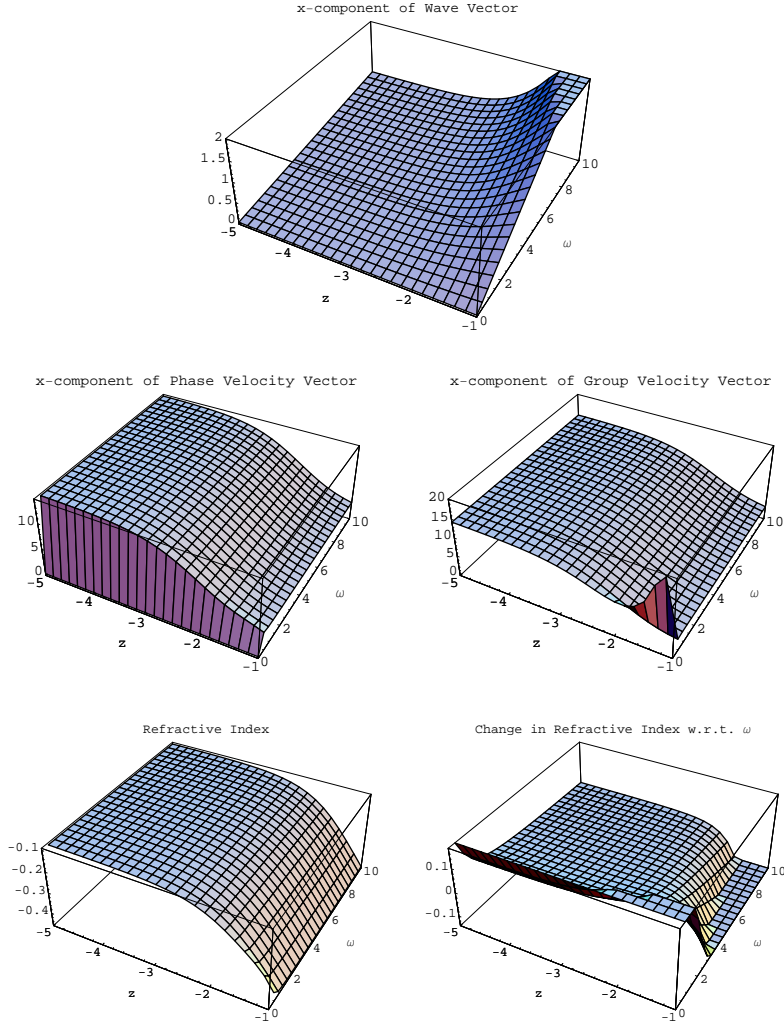


Figure 5: For the velocity components given by Eq.(5.3), Veselago medium exists in the whole region with negative phase and group velocity propagation property. The region of normal dispersion extends as the waves move away from the pair production region. The waves with negligible angular frequency do not disperse normally.

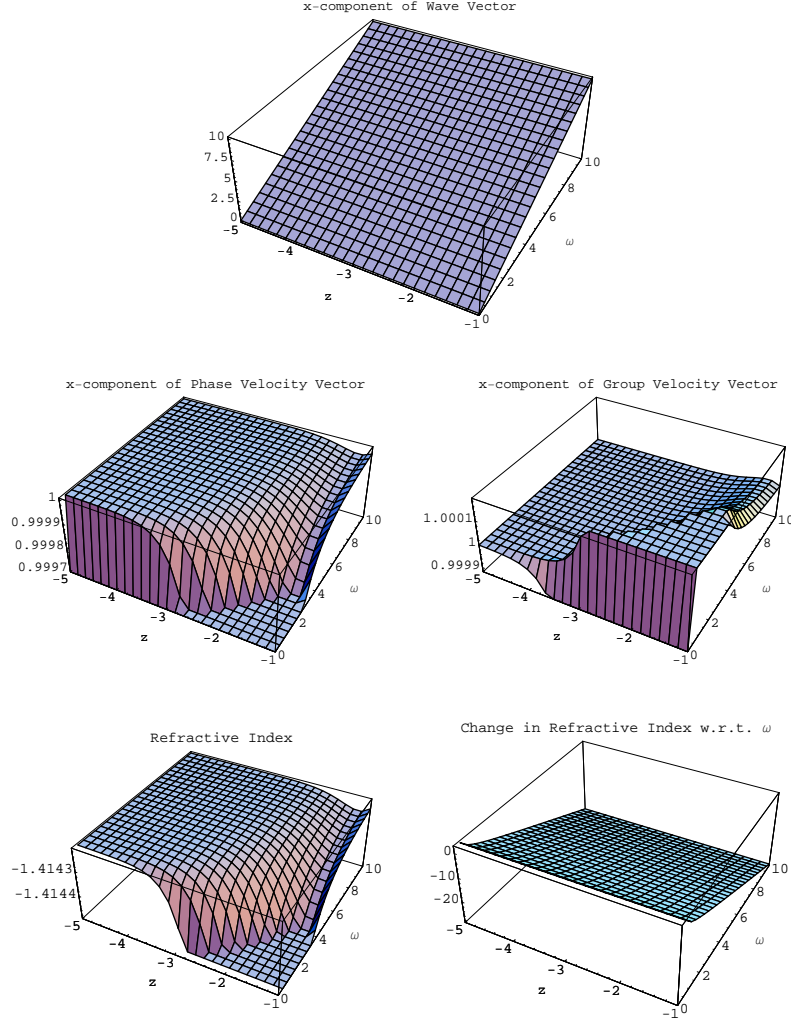


Figure 6: For the velocity components given by Eq.(5.3), plasma admits the properties of Veselago medium. The dispersion is anomalous except for the waves admitting negligible angular frequency. The negative phase and group velocity propagation regions are also observed.

energy by the Lorentz force move towards the event horizon and others move towards the outer end of the magnetosphere. These particles would reach their destinations if the plasma existing in the neighborhood of the pair production region allows them to do so. The generation of plasma is necessary to support the MHD magnetically dominated flow. Due to particle generation, waves are produced in the neighborhood of the pair production region. The dispersion relations of waves lead to understand how much the surrounding medium let the waves to disperse through.

This paper studies the wave properties for the isothermal plasma moving with velocity \mathbf{V} and admits a constant magnetic field which thread the Kerr black hole magnetosphere. The gravitomagnetic waves and the pair of particles are produced in the $z = 0$ region. If the medium living around the pair production region allows the particles and waves to pass through, the energy extraction from the black hole is possible. This can be well understood by investigating properties of the waves in this region.

We have considered a black hole immersed in a rarefied plasma with uniform magnetic field which seems to provide support for carrying currents flowing across the magnetic field lines. Due to the strength of the magnetic field, a lot of energy can be extracted due to the plasma particles that fall into the black hole's horizon has negative energy. The 3+1 GRMHD equations are derived for the neighborhood of pair production region and two-dimensional perturbations are discussed in the context of perfect MHD condition. We assume that the rotation is in the x -direction and the horizon is at $z = -\infty$. The perturbations are taken only in the xz -direction. The dispersion relations are formulated by assuming the perturbations as plane waves. We solve these relations by taking the wave vector as $(k_x, 0, -k_x)$ and obtain the x -component of the wave vector. This component leads to properties of the isothermal plasma in the neighborhood of the pair production region.

We have discussed these relations for the regions $1 \leq z \leq 5$ and $-5 \leq z \leq -1$. The dispersion relations for the region $1 \leq z \leq 5$ are shown in the Figures 1 and 2. These Figures indicate that near the pair production region, the plasma admits the properties of Veselago medium. The region which is nearer to the pair production region does not allow the waves to pass through. Thus the particles and waves cannot get out of this region. The small regions far away from the pair production region admit normal dispersion of waves which indicate that the waves pass through them. As we go away from the pair production region, normal dispersion exists frequently as shown in the Figure 1.

The region $-5 \leq z \leq -1$ allows us to investigate whether there is a possibility for the waves to move towards the black hole event horizon or not. The dispersion relations for this region are shown in the Figures 3, 4, 5 and 6. From the Figures 3, 4 and 5, we find that there are chances for the waves to pass through the neighborhood of the pair production region when the plasma admits the properties of usual or Veselago medium. The Figure 6 indicates that there can be situation when plasma admits the properties of Veselago medium in the neighborhood of the pair production region, it may not allow the waves to pass through the region.

It is interesting to note that the Figures 2 and 3 show the irregular dependence of wave vectors on the angular frequency and z . Mathematically, this irregularity is due to the nature of the roots obtained for these graphs. The irregular behavior may be due to a disturbance of the equilibrium between outward and inward directed forces. The outward directed forces are caused by the particle pressure and the curvature drift due to non-uniform magnetic field and inward directed forces are exerted by the tangential stress of the magnetic field lines for the low frequency regime.

For the high frequency regime, there is a class of MHD instabilities which sometimes develop in a thin plasma column carrying a strong current. If a kink begins to develop in such a column, the magnetic forces on the inside of the kink become larger than those on the outside, which leads to the growth of perturbation. The column then becomes unstable and causes a disruption. Both the ballooning and kink modes are ideal MHD instabilities.

In the Figures 1, 2, 5 and 6, we obtain the properties of Veselago medium. The phase and group velocity vectors propagate in the direction opposite to the Poynting vector which verify the results of Mackay et al. [22] according to which rotation of a black hole is required for the negative phase velocity propagation.

We can conclude that waves produced in the pair production region due to pair creation cannot get out of its neighborhood towards the outer end of the magnetosphere. The same result has been obtained for the cold plasma case [29]. We obtain some cases where favorable conditions are present to allow energy to move towards the black hole horizon. For the cold plasma, these conditions are present for the usual medium whereas for the isothermal plasma, these situations occur for usual as well as Veselago medium. For the plasmas with pressure, the black hole can suck particles and waves for both the usual and Veselago medium whereas for the cold plasmas, this situation holds for the usual medium.

The strong magnetic coupling enforce the accreting particles to fall into the black hole with negative energy and negative angular momentum. This indicates that energy and angular momentum flow from the black hole into the disk. When the particles fall into the black hole with negative energy, energy is extracted from the black hole [31]. When the particle with positive energy and positive angular momentum leaves the pair production region and goes towards the event horizon, much energy and momentum are lost and ultimately the particle has negative energy and angular momentum [32]. Thus if the particle either with negative or positive energy leaves the pair production region and gets a chance to reach the event horizon, the result is the extraction of energy from the black hole transmitted to the accretion disk. This shows that when the magnetosphere is filled isothermal plasma admitting the properties of Veselago as well as usual medium, our results indicate that energy extraction is possible.

Acknowledgment

We appreciate the Higher Education Commission Islamabad, Pakistan, for its financial support during this work through the *Indigenous PhD 5000 Fellowship Program Batch-II*.

Appendix A

This Appendix includes the details to reach at the perturbed form of the GRMHD equations (4.1)-(4.5). The component form of these equations is also given.

When we introduce the perturbations from Eq.(3.2), the linearized GRMHD Eqs.(4.1)-(4.5) become

$$\left\{ \left(\frac{\partial}{\partial t} - \beta \cdot \nabla \right) (\delta \mathbf{B}) \right\} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \nabla \times (\mathbf{V} \times \delta \mathbf{B}) - (\delta \mathbf{B} \cdot \nabla) \beta \quad (\text{A1})$$

$$\nabla \cdot (\delta \mathbf{B}) = 0, \quad (\text{A2})$$

$$\begin{aligned} & \left\{ \frac{\partial}{\partial t} + (\mathbf{V} - \beta) \cdot \nabla \right\} (\delta \rho + \delta p) + (\rho + p) \gamma^2 \mathbf{V} \cdot \left\{ \frac{\partial}{\partial t} + (\mathbf{V} - \beta) \cdot \nabla \right\} \mathbf{v} \\ & + (\rho + p) (\nabla \cdot \mathbf{v}) + (\delta \rho + \delta p) (\nabla \cdot \mathbf{V}) + (\delta \rho + \delta p) \gamma^2 \mathbf{V} \cdot (\mathbf{V} \cdot \nabla) \mathbf{V} \\ & = -2(\rho + p) \gamma^2 (\mathbf{V} \cdot \mathbf{v}) (\mathbf{V} \cdot \nabla) \ln \gamma - (\rho + p) \gamma^2 (\mathbf{V} \cdot \nabla \mathbf{V}) \cdot \mathbf{v} \end{aligned}$$

$$+(\rho + p)\mathbf{v} \cdot \nabla \ln u, \quad (\text{A3})$$

$$\begin{aligned} & \left[\left\{ (\rho + p)\gamma^2 + \frac{\mathbf{B}^2}{4\pi} \right\} \delta_{ij} + (\rho + p)\gamma^4 V_i V_j - \frac{1}{4\pi} B_i B_j \right] \left(\frac{\partial}{\partial t} - \beta \cdot \nabla \right) v^j \\ & + (\rho + p)\gamma^2 v_{i,j} V^j + (\rho + p)\gamma^4 V_i v_{j,k} V^j V^k + \frac{1}{4\pi} \left[\mathbf{B} \times \left\{ \mathbf{V} \times \frac{d(\delta \mathbf{B})}{d\tau} \right\} \right]_i \\ & - \frac{1}{4\pi} \{ (\delta B_i)_{,j} - (\delta B_j)_{,i} \} B^j = -(\delta p)_{,i} + \gamma^2 [(\delta \rho + \delta p) V^j \\ & + 2(\rho + p)\gamma^2 (\mathbf{V} \cdot \mathbf{v}) V^j + (\rho + p)v^j] \beta_{j,i} + \frac{1}{4\pi} (B_{i,j} - B_{j,i}) \delta B^j \\ & - (\rho + p)\gamma^4 (v_i V^j + v_j V^i) V_{k,j} V^k \\ & - \gamma^2 \{ (\delta \rho + \delta p) V^j + 2(\rho + p)\gamma^2 (\mathbf{V} \cdot \mathbf{v}) V^j + (\rho + p)v^j \} V_{i,j} \\ & - \gamma^4 V_i \{ (\delta \rho + \delta p) V^j + 4(\rho + p)\gamma^2 (\mathbf{V} \cdot \mathbf{v}) V^j + (\rho + p)v^j \} V_{j,k} V^k, \quad (\text{A4}) \\ & \gamma^2 \left\{ \frac{\partial}{\partial t} + (\mathbf{V} - \beta) \cdot \nabla \right\} (\delta \rho + \delta p) + 2(\rho + p)\gamma^4 \mathbf{V} \cdot \left\{ \frac{\partial}{\partial t} + (\mathbf{V} - \beta) \cdot \nabla \right\} \mathbf{v} \\ & - (\rho + p)\gamma^2 \mathbf{v} \cdot \nabla \ln u + (\rho + p)\gamma^4 \mathbf{V} \cdot (\mathbf{v} \cdot \nabla) \mathbf{V} + 6(\rho + p)\gamma^6 (\mathbf{V} \cdot \mathbf{v}) \mathbf{V} \cdot (\mathbf{V} \cdot \nabla) \mathbf{V} \\ & + 2(\delta \rho + \delta p)\gamma^4 \mathbf{V} \cdot (\mathbf{V} \cdot \nabla) \mathbf{V} + 2(\rho + p)\gamma^4 \mathbf{v} \cdot (\mathbf{V} \cdot \nabla) \mathbf{V} \\ & - 2(\rho + p)\gamma^4 (\mathbf{V} \cdot \mathbf{v}) \mathbf{V} \cdot \nabla \ln u + 2(\rho + p)\gamma^4 (\mathbf{V} \cdot \mathbf{v}) (\nabla \cdot \mathbf{V}) - \frac{\partial}{\partial t} (\delta p) \\ & + (\delta \rho + \delta p)\gamma^2 (\nabla \cdot \mathbf{V}) + (\rho + p)\gamma^2 (\nabla \cdot \mathbf{v}) - \gamma^2 (\beta \cdot \nabla) (\delta \rho + \delta p) \\ & + 2(\rho + p)\gamma^4 (\mathbf{V} \cdot \mathbf{v}) (\beta \cdot \nabla \ln u) - 6(\rho + p)\gamma^6 (\mathbf{V} \cdot \mathbf{v}) \mathbf{V} \cdot (\beta \cdot \nabla) \mathbf{V} \\ & - 2(\rho + p)\gamma^4 \mathbf{v} \cdot (\beta \cdot \nabla) \mathbf{V} - 2(\delta \rho + \delta p)\gamma^4 \mathbf{V} \cdot (\beta \cdot \nabla) \mathbf{V} \\ & - (\delta \rho + \delta p)\gamma^2 \mathbf{V} \cdot (\mathbf{V} \cdot \nabla) \beta - (\rho + p)\gamma^2 \mathbf{V} \cdot (\mathbf{v} \cdot \nabla) \beta \\ & - 2(\rho + p)\gamma^4 (\mathbf{V} \cdot \mathbf{v}) \mathbf{V} \cdot (\mathbf{V} \cdot \nabla) \beta + \frac{1}{4\pi} [(\mathbf{v} \times \mathbf{B}) \cdot (\nabla \times \mathbf{B}) \\ & + (\mathbf{V} \times \delta \mathbf{B}) \cdot (\nabla \times \mathbf{B}) + (\mathbf{V} \times \mathbf{B}) \cdot (\nabla \times \delta \mathbf{B}) \\ & + (\mathbf{V} \times \mathbf{B}) \cdot \left\{ \frac{d\mathbf{v}}{d\tau} \times \mathbf{B} + \mathbf{v} \times \frac{d\delta \mathbf{B}}{d\tau} \right\}] = 0. \quad (\text{A5}) \end{aligned}$$

The component form of these equations are

$$\frac{db_x}{d\tau} + V b_{x,x} + u b_{x,z} = -u' b_x + (V - \beta)' b_z + v_{x,z} - \lambda v_{z,z} - \lambda' v_z, \quad (\text{A6})$$

$$\frac{db_z}{d\tau} + V b_{z,x} + u b_{z,z} = \lambda v_{z,x} - v_{x,x}, \quad (\text{A7})$$

$$b_{x,x} + b_{z,z} = 0, \quad (\text{A8})$$

$$\begin{aligned}
& \rho \frac{d\tilde{\rho}}{d\tau} + p \frac{d\tilde{p}}{d\tau} + \rho V \tilde{\rho}_{,x} + p V \tilde{p}_{,x} + \rho u \tilde{\rho}_{,z} + p u \tilde{p}_{,z} - (\tilde{\rho} - \tilde{p}) \{p' u + p u'\} \\
& + p u \gamma^2 (V V' + u u') \} + (\rho + p) \gamma^2 \left(V \frac{dv_x}{d\tau} + u \frac{dv_z}{d\tau} \right) + (\rho + p) (1 + \gamma^2 V^2) v_{x,x} \\
& + (\rho + p) (1 + \gamma^2 u^2) v_{z,z} + (\rho + p) u V \gamma^2 (v_{x,z} + v_{z,x}) \\
& = -(\rho + p) \gamma^2 u [(1 + 2\gamma^2 V^2) V' + 2\gamma^2 u V u'] v_x \\
& + (\rho + p) [(1 - 2\gamma^2 u^2) (1 + \gamma^2 u^2) \frac{u'}{u} - 2\gamma^4 u^2 V V'] v_z, \tag{A9}
\end{aligned}$$

$$\begin{aligned}
& \left\{ (\rho + p) \gamma^2 (1 + \gamma^2 V^2) + \frac{B^2}{4\pi} \right\} \frac{dv_x}{d\tau} + \left\{ (\rho + p) \gamma^4 u V - \frac{\lambda B^2}{4\pi} \right\} \frac{dv_z}{d\tau} \\
& + \left\{ (\rho + p) \gamma^2 (1 + \gamma^2 V^2) - \frac{B^2}{4\pi} \right\} (V v_{x,x} + u v_{x,z}) - \frac{B^2}{2\pi} u V b_{z,z} \\
& + \left\{ (\rho + p) \gamma^4 u V + \frac{\lambda B^2}{4\pi} \right\} (V v_{z,x} + u v_{z,z}) + \frac{B^2}{4\pi} \{ (1 - V^2) b_{z,x} \\
& - (1 - u^2) b_{x,z} \} = -\frac{B^2}{4\pi} u u' b_x + \frac{B^2}{4\pi} \{ \lambda' + u(V - \beta)' \} b_z - p \tilde{p}_{,x} \\
& - (\rho \tilde{\rho} + p \tilde{p}) \gamma^2 u \{ (1 + \gamma^2 V^2) V' + \gamma^2 u V u' \} - (\rho + p) \gamma^4 u \{ (1 + 4\gamma^2 V^2) u u' \\
& + 4(1 + \gamma^2 V^2) V V' \} v_x - [(\rho + p) \gamma^2 \{ (1 + 2\gamma^2 u^2) (1 + 2\gamma^2 V^2) - \gamma^2 V^2 \} V' \\
& + 2\gamma^2 (1 + 2\gamma^2 u^2) u V u'] + \frac{B^2}{4\pi} u \lambda' v_z, \tag{A10}
\end{aligned}$$

$$\begin{aligned}
& \left\{ (\rho + p) \gamma^2 (1 + \gamma^2 u^2) + \frac{\lambda^2 B^2}{4\pi} \right\} \frac{dv_z}{d\tau} + \left\{ (\rho + p) \gamma^4 u V - \frac{\lambda B^2}{4\pi} \right\} \frac{dv_x}{d\tau} \\
& + \left\{ (\rho + p) \gamma^2 (1 + \gamma^2 u^2) - \frac{\lambda^2 B^2}{4\pi} \right\} (V v_{z,x} + u v_{z,z}) + \frac{\lambda B^2}{2\pi} u V b_{z,z} \\
& + \left\{ (\rho + p) \gamma^4 u V + \frac{\lambda B^2}{4\pi} \right\} (V v_{x,x} + u v_{x,z}) - \frac{\lambda B^2}{4\pi} \{ (1 - V^2) b_{z,x} \\
& - (1 - u^2) b_{x,z} \} = -\frac{B^2}{4\pi} (\lambda' - \lambda u u') b_x - \frac{\lambda B^2}{4\pi} u (V - \beta)' b_z \\
& - \rho \tilde{\rho} \gamma^2 \{ u u' (1 + \gamma^2 u^2) + \gamma^2 u^2 V V' - V \beta' \} - [p \tilde{p}_{,z} + p' \tilde{p} \\
& + p \tilde{p} \gamma^2 \{ u u' (1 + \gamma^2 u^2) + \gamma^2 u^2 V V' - V \beta' \}] - (\rho + p) \gamma^2 [\gamma^2 u^2 (1 + 4\gamma^2 V^2) V' \\
& - (1 + 2\gamma^2 V^2) \beta' + 2\gamma^2 u V (1 + 2\gamma^2 u^2) u'] v_x - [(\rho + p) \gamma^2 \{ -2\gamma^2 u V \beta' \\
& + (1 + \gamma^2 u^2) (1 + 4\gamma^2 u^2) u' + 2\gamma^2 (1 + 2\gamma^2 u^2) u V V' \} - \frac{\lambda B^2}{4\pi} u \lambda' v_z], \tag{A11}
\end{aligned}$$

$$\gamma^2 \rho \frac{\partial \tilde{\rho}}{\partial t} + p (\gamma^2 - 1) \frac{\partial \tilde{p}}{\partial t} + \tilde{\rho} \gamma^2 \{ \rho' u + \rho u' + 2\rho u \gamma^2 (V V' + u u') \}$$

$$\begin{aligned}
& -\rho u V \beta'\} + \tilde{p} \gamma^2 \{u p' + u' p + 2 p u \gamma^2 (V V' + u u') - p \gamma^2 u V \beta'\} \\
& + \gamma^2 \rho \tilde{\rho}_{,x} (V - \beta) + \gamma^2 u \rho \tilde{\rho}_{,z} + \gamma^2 p \tilde{p}_{,x} (V - \beta) + \gamma^2 u p \tilde{p}_{,z} \\
& + \frac{\partial v_x}{\partial t} \{2(\rho + p) \gamma^4 V - \frac{B^2}{4\pi} (u \lambda - V)\} + \frac{\partial v_z}{\partial t} \{2(\rho + p) \gamma^4 u + \frac{\lambda B^2}{4\pi} (u \lambda - V)\} \\
& + v_{x,x} [(\rho + p) \gamma^2 \{1 + 2 \gamma^2 V (V - \beta)\} - \frac{B^2}{4\pi} (V - \beta) (u \lambda - V)] \\
& + v_{x,z} u [2(\rho + p) \gamma^4 V + \frac{B^2}{4\pi} (u \lambda - V)] + v_{z,x} (V - \beta) [2(\rho + p) \gamma^4 u \\
& + \frac{B^2 \lambda}{4\pi} (u \lambda - V)] + v_{z,z} \{(\rho + p) (1 + 2 \gamma^2 u^2) - \frac{B^2 \lambda}{4\pi} u (u \lambda - V)\} \\
& + \frac{B^2}{4\pi} (u \lambda - V) \{(1 - u^2) b_{x,z} - (1 - V^2) b_{z,x} + 2 u V b_{z,z}\} \\
& + \frac{B^2}{4\pi} u b_x \{\lambda' - (u \lambda - V) u'\} + \frac{B^2}{4\pi} b_z \{-\lambda' V + u (u \lambda - V) (V - \beta)'\} \\
& + v_x [(\rho + p) \gamma^2 u \{2 \gamma^2 V' + 6 \gamma^4 V (V V' + u u') - \beta' (1 + 2 \gamma^2 V^2)\} - \frac{B^2 \lambda'}{4\pi}] \\
& + v_z [\frac{B^2 \lambda'}{4\pi} \{\lambda - u (u \lambda - V)\} + (\rho + p) \gamma^2 \{-\frac{u'}{u} + 2 \gamma^2 u u' + 6 \gamma^4 u^2 (V V' + u u') \\
& + \gamma^2 (V V' + u u') - V \beta' (1 + \gamma^2 u^2)\}] = 0. \tag{A12}
\end{aligned}$$

We have used the conservation law of rest-mass for three-dimensional hypersurface, i.e., given by Eq.(5.13) to simplify Eq.(A9). The same equation will be used to simplify the Fourier analyzed form of Eqs (4.11)-(4.17).

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